

Equivariant Poincaré series of filtrations ^{*}

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Abstract

We offer a new approach to a definition of an equivariant version of the Poincaré series. This Poincaré series is defined not as a power series, but as an element of the Grothendieck ring of G -sets with an additional structure. We compute this Poincaré series for natural filtrations on the ring of germs of functions on the plane $(\mathbb{C}^2, 0)$ with a finite group representation.

Introduction

The notion of the Poincaré series of a multi-index filtration (say on the ring of germs of functions on a variety) was introduced in [4]. It can be expressed as an integral with respect to the Euler characteristic (duly defined) over the projectivization of the space of germs of functions (see e.g. [1]). For some natural multi-index filtrations on the ring of germs of functions the Poincaré series are related with topological invariants of singularities. For example the Poincaré series of the multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^2, 0}$ of germs of functions in two variables defined by orders of a function on branches of a plane curve singularity tends to coincide with the Alexander polynomial (in several variables) of the corresponding link: see e.g. [1]. An equivariant version of the Poincaré series (for an action of a finite group G) was defined in [2]. For universal Abelian covers of rational surface singularities the equivariant Poincaré series was computed in [3]. A. Némethi has found that the equivariant Poincaré series of Abelian covers of surface singularities are connected with computation of Seiberg–Witten invariants of their links: [7]. The equivariant Poincaré series in [2] is defined as an integral with respect to the Euler characteristic over

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the projectivization of the union of subspaces of equivariant functions (with respect to one-dimensional representations of the group). Thus it takes into account only one-dimensional representations ignoring all others. Therefore it seems to be better adjusted for Abelian groups. Nevertheless even in this case it ignores some important information. For example it does not define, in general, the usual (non-equivariant) Poincaré series.

Here we offer a new approach to a definition of an equivariant version of the Poincaré series. This Poincaré series is defined not as a power series, but as an element of the Grothendieck ring of G -sets with an additional structure. (This Grothendieck ring, in a natural way, is isomorphic to the series ring $\mathbb{Z}[[t_1, \dots, t_r]]$ for the case of the trivial group.) The idea to define equivariant analogues of (numerical) invariants as elements of the Grothendieck ring of G -sets was used, for example, in [6, 5] where an equivariant analogue of the Lefschetz number was defined as an element of the Burnside ring which coincides with the Grothendieck ring of finite G -sets.

We compute the equivariant Poincaré series for natural filtrations on the ring of germs of functions on the plane $(\mathbb{C}^2, 0)$ with a finite group representation. In this cases the expressions for the equivariant Poincaré series are of A'Campo type, i.e. this Poincaré series are products/ratios of binomials of the form $(1 - U)$ for some “irreducible” elements U .

1 Equivariant Poincaré series

Let $(V, 0)$ be a germ of a complex analytic variety with an action of a finite group G . The group G acts on the ring $\mathcal{O}_{V,0}$ of germs of functions on $(V, 0)$: $a^*f(x) = f(a^{-1}x)$ ($f \in \mathcal{O}_{V,0}$, $a \in G$, $x \in V$). A function $v : \mathcal{O}_{V,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is called an *order function* if $v(\lambda f) = v(f)$ for a non-zero $\lambda \in \mathbb{C}$ and $v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}$. (If besides that $v(f_1 f_2) = v(f_1) + v(f_2)$, the function v is a valuation.) A multi-index filtration of the ring $\mathcal{O}_{V,0}$ is defined by a collection v_1, \dots, v_r of order functions:

$$J(\underline{v}) = \{f \in \mathcal{O}_{V,0} : \underline{v}(f) \geq \underline{v}\}, \quad (1)$$

where $\underline{v} = (v_1, \dots, v_r) \in \mathbb{Z}_{\geq 0}^r$, $\underline{v}(f) = (v_1(f), \dots, v_r(f))$ and $(v'_1, \dots, v'_r) \geq (v''_1, \dots, v''_r)$ if and only if $v'_i \geq v''_i$ for all $i = 1, \dots, r$.

Let us assume that the filtration $J(\underline{v})$ is *finitely determined*. This means that, for any $\underline{v} \in \mathbb{Z}_{\geq 0}^r$, there exists an integer k such that $\mathfrak{m}^k \subset J(\underline{v})$ where \mathfrak{m} is the maximal ideal in $\mathcal{O}_{V,0}$. Let $\mathbb{P}\mathcal{O}_{V,0}$ be the projectivization of the ring $\mathcal{O}_{V,0}$. For $k \geq 0$, let $J_{V,0}^k = \mathcal{O}_{V,0}/\mathfrak{m}^{k+1}$ be the space of k -jets of functions. It is a finite-dimensional vector space. Let $\mathbb{P}J_{V,0}^k$ be the projectivization of the

jet space and let $\mathbb{P}^*J_{V,0}^k = \mathbb{P}J_{V,0}^k \cup \{*\}$. Here one can say that $\mathbb{P}^*J_{V,0}^k$ is the factor space of $J_{V,0}^k$ by the standar \mathbb{C}^* -action: the added point $*$ corresponds to the orbit of the origin. There are natural maps $\pi_k : \mathbb{P}\mathcal{O}_{V,0} \rightarrow \mathbb{P}^*J_{V,0}^k$ and $\pi_{k,\ell} : \mathbb{P}J_{V,0}^k \rightarrow \mathbb{P}^*J_{V,0}^\ell$ for $k \geq \ell$. A subset $B \subset \mathbb{P}\mathcal{O}_{V,0}$ is called *cylindric* if there exists k and a constructible subset $X \subset \mathbb{P}J_{V,0}^k$ ($\subset \mathbb{P}^*J_{V,0}^k$) such that $B = \pi_k^{-1}(X)$. The Euler characteristic $\chi(B)$ of a cylindric subset $B = \pi_k^{-1}(X)$ is defined as $\chi(X)$. A function ψ on $\mathbb{P}\mathcal{O}_{V,0}$ with values in an Abelian group A is called *cylindric* if, for any $a \in A$, $a \neq 0$, $\psi^{-1}(a)$ is cylindric. The integral of a cylindric function $\psi : \mathbb{P}\mathcal{O}_{V,0} \rightarrow A$ with respect to the Euler characteristic is defined as

$$\int_{\mathbb{P}\mathcal{O}_{V,0}} \psi d\chi = \sum_{a \in A, a \neq 0} \chi(\psi^{-1}(a))a$$

(if the right hand side makes sense in the group A ; otherwise the function ψ is not integrable).

In the same way one can define cylindric sets, their Euler characteristics and the integral with respect to the Euler characteristic for the factor $\mathbb{P}\mathcal{O}_{V,0}/G$ of the space $\mathbb{P}\mathcal{O}_{V,0}$ by an action of a finite group G which respects the filtration by the powers of the maximal ideal.

The (usual) Poincaré series of the multi-index filtration (1) can be defined as

$$P_{\{v_i\}}(t_1, \dots, t_r) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{v}(f)} d\chi,$$

where $\underline{t} = (t_1, \dots, t_r)$, $\underline{t}^{\underline{v}} = t_1^{v_1} \cdot \dots \cdot t_r^{v_r}$; $\underline{t}^{\underline{v}(f)}$ is considered as a function on $\mathbb{P}\mathcal{O}_{V,0}$ with values in the ring (Abelian group) $\mathbb{Z}[[t_1, \dots, t_r]]$, $t^{+\infty}$ is assumed to be equal to zero.

We shall define an equivariant version of the Poincaré series of the filtration (1) as an element of the Grothendieck ring of sets with an additional structure.

Definition: A (“locally finite”) (G, r) – set A is a triple $(X^A, \underline{w}^A, \alpha^A)$ where

- X^A is a G -set, i.e. a set with a G -action;
- \underline{w}^A is a function on X^A with values in $\mathbb{Z}_{\geq 0}^r$;
- α^A associates to each point $x \in X^A$ a one-dimensional representation α_x^A of the isotropy subgroup $G_x = \{a \in G : ax = x\}$ of the point x ;

satisfying the following conditions:

- 1) $\alpha_{ax}^A = a\alpha_x^A a^{-1}$ for $x \in X^A$, $a \in G$;
- 2) for any $\underline{w} \in \mathbb{Z}_{\geq 0}^r$ the set $\{x \in X^A : \underline{w}^A(x) \leq \underline{w}\}$ is finite.

All (locally finite) (G, r) -sets form an Abelian semigroup in which the sum is defined as the disjoint union. Let the Cartesian product $A \times B$ of two (G, r) -sets A and B be the triple where $X^{A \times B} = X^A \times X^B$ with the natural action of the group G , $\underline{w}^{A \times B}(x, y) = \underline{w}^A(x) + \underline{w}^B(y)$, $\alpha_{(x, y)}^{A \times B} = \alpha_x^A \cdot \alpha_y^B$. This makes the semigroup of (G, r) -sets a semiring. The unit 1 is represented by the one-point set $X^1 = \{*\}$ with $\underline{w}^1(*) = 0$ and $\alpha_*^1 = 1$.

Let $K_0((G, r) - \text{sets})$ be the Grothendieck ring of (locally finite) (G, r) -sets, i.e. the Grothendieck group corresponding to the semigroup described above with the multiplication defined by the Cartesian product.

Example. For the trivial group $G = \{e\}$, the ring $K_0((G, r) - \text{sets})$ is isomorphic to the ring $\mathbb{Z}[[t_1, \dots, t_r]]$ of power series where t_i is the one-point set $X^{t_i} = \{*\}$ with $\underline{w}^{t_i}(*) = (0, \dots, 1, \dots, 0)$, 1 is at the i -th place.

Remarks.

1. Let K'_0 be the subring of the Grothendieck ring $K_0((G, r) - \text{sets})$ generated by elements with the trivial action of the group G . Let $R(G)$ be the ring of representations of the group G and let $R_1(G)$ be its subring generated by one-dimensional representations. The ring K'_0 in a natural way is isomorphic to the ring $R_1(G)[[t_1, \dots, t_r]]$ of power series with coefficients in the ring $R_1(G)$. This is just the ring where the equivariant Poincaré series defined in [2] lives.

2. For an arbitrary group G , one can choose a system of elements which generates (after completion) the ring $K_0((G, r) - \text{sets})$. This identifies the Grothendieck ring $K_0((G, r) - \text{sets})$ with the ring of series in several variables modulo certain relations. However this is somewhat artificial.

Example. Let $G = \mathbb{Z}_2$ and let $r = 1$. A natural set of (multiplicative) generators of the Grothendieck ring $K_0((G, r) - \text{sets})$ consists of the following elements:

1. t_1 : $X^{t_1} = \mathbb{Z}_2$, the function w^{t_1} takes two values 0 and 1 (the isotropy subgroup of a point is trivial and thus $\alpha_x^{t_1} = 1$);
2. t_2 : $X^{t_2} = \mathbb{Z}_2$, the function w^{t_2} takes value 0 at each point;
3. t_3 : $X^{t_3} = \{*\} = \mathbb{Z}_2/\mathbb{Z}_2$, $w^{t_3}(*) = 1$, $\alpha_*^{t_3} = 1$;
4. t_4 : $X^{t_4} = \{*\} = \mathbb{Z}_2/\mathbb{Z}_2$, $w^{t_4}(*) = 0$, $\alpha_*^{t_4} = -1$;

One can see that $t_4^2 = 1$, $t_2^2 = 2t_2$, $t_2t_4 = t_2$, $t_1t_4 = t_1$, $t_1t_2 = 2t_1$ and $K_0((G, r) - \text{sets}) = \mathbb{Z}[[t_1, t_2, t_3, t_4]]/\langle t_4^2 - 1, t_2^2 - 2t_2, t_2t_4 - t_2, t_1t_4 - t_1, t_1t_2 - 2t_1 \rangle$. Pay attention that natural additive generators of $K_0((G, r) - \text{sets})$ like $A = A_{m_1m_2}$: $X^A = \mathbb{Z}_2$, w^A takes the values m_1 and m_2 ($\alpha_x^A = 1$ for each x) have

somewhat complicated expressions in terms of the multiplicative generators t_i , $i = 1, \dots, 4$. For example $A_{15} = t_1^4 t_3 + t_2 t_3^3 - 4t_1^2 t_3^2$. The ring $K_0((G, r) - \text{sets})$ can be considered as a completion of the ring of regular functions (polynomials) on the union $\mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^1$ (the variety defined by the ideal $\langle t_4^2 - 1, t_2^2 - 2t_2, t_2 t_4 - t_2, t_1 t_4 - t_1, t_1 t_2 - 2t_1 \rangle$).

Let again the filtration $\{J(\underline{v})\}$ be defined by the order functions v_1, \dots, v_r by (1). For an element $f \in \mathbb{P}\mathcal{O}_{V,0}$, that is for a function germ defined up to a constant factor, let G_f be the isotropy subgroup of the corresponding point of $\mathbb{P}\mathcal{O}_{V,0}$: $G_f = \{a \in G : a^* f = \lambda_f(a) f\}$. The map $a \mapsto \lambda_f(a)$ defines a one-dimensional representation λ_f of the subgroup G_f . (The representation λ_f is defined by the class of f in the projectivization $\mathbb{P}\mathcal{O}_{V,0}$.) For an element $f \in \mathbb{P}\mathcal{O}_{V,0}$ (or rather for its orbit Gf), let T_f be the element of the Grothendieck ring $K_0((G, r) - \text{sets})$ represented by the orbit Gf of f (as a G -set) with $\underline{w}^{T_f}(a^* f) = \underline{v}(a^* f)$ and $\alpha_{a^* f}^{T_f} = \lambda_{a^* f}$ ($a \in G$). One should exclude those f for which $\underline{w}^{T_f}(a^* f) \notin \mathbb{Z}_{\geq 0}^r$, i.e. $w_i^{T_f}(a^* f) = +\infty$ for some $a \in G$. One can formally define T_f to be equal to zero in this case. (This is an analogue of the assumption $t^{+\infty} = 0$ in the definition of the usual Poincaré series.)

Let us consider $T : f \mapsto T_f$ as a function on $\mathbb{P}\mathcal{O}_{V,0}/G$ with values in the Grothendieck ring $K_0((G, r) - \text{sets})$. One can see that this function is cylindric. This follows from the condition that the filtration $\{J(\underline{v})\}$ is finitely determined. Moreover, the function T is obviously integrable (with respect to the Euler characteristic).

Definition: The equivariant Poincaré series $P_{\{v_i\}}^G$ of the filtration $\{J(\underline{v})\}$ is defined by

$$P_{\{v_i\}}^G = \int_{\mathbb{P}\mathcal{O}_{V,0}/G} T_f d\chi \in K_0((G, r) - \text{sets}) .$$

Example. Let $F_G \subset \mathbb{P}\mathcal{O}_{V,0}$ be the set of fixed points of the action of the group G on $\mathbb{P}\mathcal{O}_{V,0}$. One has $F_G/G \cong F_G$ and F_G consists of classes of functions equivariant with respect to the G -action: $a^* f = \lambda_f(a) f$. For $f \in F_G$, $T_f \in R_1(G)[[t_1, \dots, t_r]]$ (see Remark 1 above). One can see that the integral $\int_{F_G} T_f d\chi$ as an element of $R_1(G)[[t_1, \dots, t_r]]$ coincides with the equivariant Poincaré series $P_{\{v_i\}}^G(t_1, \dots, t_r)$ in the sense of [2].

Statement 1 *The equivariant Poincaré series $P_{\{v_i\}}^G$ determines the equivariant Poincaré series $P_{\{v_i\}}^G(t_1, \dots, t_r)$ in the sense of [2].*

Proof. One can see that the Grothendieck ring $K_0((G, r) - \text{sets})$ is a semidirect product of two subrings $K'_0 \simeq R_1(G)[[t_1, \dots, t_r]]$ and K''_0 generated by elements

represented by G -sets with the trivial action of the group G and with actions without fixed points respectively. Let $\Pi' : K_0((G, r) - \text{sets}) \rightarrow K'_0$ be the projection onto the first summand. One can see that $P_{\{v_i\}}^G(t_1, \dots, t_r) = \Pi'(P_{\{v_i\}}^G)$. \square

Suppose that all the order functions v_i are such that $v_i(f) = +\infty$ if and only if $f = 0$. This takes place, for example, if all v_i are divisorial valuations.

Statement 2 *In the described situation the equivariant Poincaré series $P_{\{v_i\}}^G$ determines the (usual) Poincaré series $P_{\{v_i\}}(t_1, \dots, t_r) \in \mathbb{Z}[[t_1, \dots, t_r]]$.*

Proof. Let $\Pi : K_0((G, r) - \text{sets}) \rightarrow \mathbb{Z}[[t_1, \dots, t_r]]$ be the homomorphism which sends an element A of $K_0((G, r) - \text{sets})$ to $\sum_{x \in X^A} \underline{t}^{w^A(x)}$. The Fubini formula (see e.g. [1]) applied to the factorization map $p : \mathbb{P}\mathcal{O}_{V,0} \rightarrow \mathbb{P}\mathcal{O}_{V,0}/G$ gives

$$P_{\{v_i\}}(t_1, \dots, t_r) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^v d\chi = \int_{\mathbb{P}\mathcal{O}_{V,0}/G} \Pi(T_f) d\chi = \Pi(P_{\{v_i\}}^G). \quad (2)$$

\square

Remark. Another situation when Equation 2 (and thus Statement 2) holds is when the set $\{v_i\}$ is G -invariant, i.e. contains all shifts v_{ig} of its elements v_i for $g \in G$: $v_{ig}(f) := v_i((g^{-1})^*f)$.

2 The equivariant Poincaré series for filtrations on $\mathcal{O}_{\mathbb{C}^2,0}$

Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$ be a modification of the plane $(\mathbb{C}^2, 0)$ by a G -invariant sequence of blowing-ups. This means that the action of G on the plane \mathbb{C}^2 lifts to an action on the smooth surface \mathcal{X} and the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ (a normal crossing divisor on \mathcal{X}) is invariant with respect to the G -action. At each intersection point x of two components of the exceptional divisor \mathcal{D} , each of these components is invariant with respect to the isotropy subgroup $G_x = \{a \in G : ax = x\}$ of the point x . All components E_σ ($\sigma \in \Gamma$) of the exceptional divisor \mathcal{D} are isomorphic to the complex projective line \mathbb{CP}^1 . Let \dot{E}_σ be the “smooth part” of the component E_σ , i.e. E_σ itself minus intersection points with all other components of the exceptional divisor \mathcal{D} , let $\dot{\mathcal{D}} = \bigcup_{\sigma} \dot{E}_\sigma$ be the smooth part of the exceptional divisor \mathcal{D} , and let $\widehat{\mathcal{D}} = \dot{\mathcal{D}}/G$ be the

corresponding factor space, i.e. the space of orbits of the action of the group G on $\dot{\mathcal{D}}$. Let $p : \dot{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$ be the factorization map.

For $x \in \dot{\mathcal{D}}$, let \widetilde{L}_x be a germ of a smooth curve on \mathcal{X} transversal to $\dot{\mathcal{D}}$ at the point x and invariant with respect to the isotropy subgroup G_x of the point x . The image $L_x = \pi(\widetilde{L}_x) \subset (\mathbb{C}^2, 0)$ is called a *curvette* at the point x . Let the curvette L_x be given by an equation $h'_x = 0$, $h'_x \in \mathcal{O}_{\mathbb{C}^2, 0}$. Let $h_x = \sum_{a \in G_x} \frac{h'_x}{a^* h'_x}(0) \cdot a^* h'_x$. The germ h_x is G_x -equivariant and $\{h_x = 0\} = L_x$. Moreover, in what follows we assume that the germ h_x is fixed this way for one point x of each G -orbit and is defined by $h_{ax} = ah_x a^{-1}$ for other points of the orbit.

Let $\{\Xi\}$ be a stratification of the space (in fact of a smooth curve) $\widehat{\mathcal{D}}$ ($\widehat{\mathcal{D}} = \coprod \Xi$) such that:

- 1) each stratum Ξ is connected;
- 2) for each point $\widehat{x} \in \Xi$ and for each point x from its preimage $p^{-1}(\widehat{x})$, the conjugacy class of the isotropy subgroup G_x of the point x is the same, i.e. depends only on the stratum Ξ .

The last is equivalent to say that, over each stratum Ξ , the map $p : \dot{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$ is a covering.

For a component E_σ of the exceptional divisor \mathcal{D} , let v_σ be the corresponding divisorial valuation on the ring $\mathcal{O}_{\mathbb{C}^2, 0}$: for $f \in \mathcal{O}_{\mathbb{C}^2, 0}$, $v_\sigma(f)$ is the order of zero of the lifting $f \circ \pi$ of the function f along the component E_σ . Let $\{1, \dots, r\}$ be a subset of Γ , and let v_1, \dots, v_r be the corresponding divisorial valuations. They define the multi-index filtration (1).

For a point $x \in \dot{\mathcal{D}}$, let T_x be the element of the Grothendieck ring $K_0((G, r)\text{-sets})$ defined by $T_x = T_{h_x}$ where h_x is a G_x -equivariant function defining a curvette at the point x . The element T_x is well-defined. i.e. does not depend on the choice of the function h_x . One can see that the element T_x is one and the same for all points from the preimage of a stratum Ξ and therefore it will be denoted by T_Ξ .

Theorem 1

$$P_{\{v_i\}}^G = \prod_{\{\Xi\}} (1 - T_\Xi)^{-\chi(\Xi)}. \quad (3)$$

Proof. Let Y be the configuration space of effective divisors on $\dot{\mathcal{D}}$. The space

Y has the natural representation of the form

$$Y = \coprod_{\{k_\sigma\}} \left(\prod_{\sigma \in \Gamma} S^{k_\sigma} \dot{E}_\sigma \right) = \prod_{\sigma \in \Gamma} \left(\prod_{k=0}^{\infty} S^k \dot{E}_\sigma \right),$$

where $S^k Z = Z^k / S_k$ is the k -th symmetric power of the space Z . There is the natural action of the group G on the space Y . Let $\hat{Y} = Y/G$ be the space of G -orbits on Y .

For a point $y \in Y$, $y = \sum_{i=1}^n x_i$, let T_y be the element of the Grothendieck ring $K_0((G, r)\text{-sets})$ defined by $T(y) = T_{\prod h_{x_i}}$. This way one has a G -invariant map $T : Y \rightarrow K_0((G, r)\text{-sets})$ and therefore a map $\hat{T} : \hat{Y} \rightarrow K_0((G, r)\text{-sets})$.

Let \tilde{Y} be the configuration space of effective divisors on the (smooth) curve $\hat{\mathcal{D}} = \dot{\mathcal{D}}/G$. The space \tilde{Y} has the natural representation (corresponding to the stratification $\{\Xi\}$) of the form

$$\tilde{Y} = \coprod_{\{k_\Xi\}} \left(\prod_{\Xi} S^{k_\Xi} \Xi \right) = \prod_{\Xi} \left(\prod_{k=0}^{\infty} S^k \Xi \right).$$

Let $\tilde{T} : \tilde{Y} \rightarrow K_0((G, r)\text{-sets})$ be the function on \tilde{Y} defined by

$$\tilde{T}(\tilde{y}) = \prod_{\Xi} T_{\Xi}^{k_\Xi}$$

for $\tilde{y} \in \prod_{\Xi} S^{k_\Xi} \Xi$.

There is the natural map $q : \hat{Y} \rightarrow \tilde{Y}$ which sends the orbit of a point $y = \sum_{i=1}^m x_i \in Y$ to the point $\tilde{y} = \sum_{i=1}^m \tilde{x}_i$, where $\tilde{x}_i = p(x_i)$ is the orbit of the point x_i . The preimage of a point $\tilde{y} = \sum_{i=1}^m \tilde{x}_i \in \tilde{Y}$ in Y (i.e. under the map $q \circ p$) is the product of the orbits Gx_i , $p(x_i) = \tilde{x}_i$, with the natural action of the group G . Moreover $\tilde{T}(\tilde{y}) = \sum_{\hat{y} \in q^{-1}(\tilde{y})} \hat{T}(\hat{y})$.

Therefore

$$\int_{\hat{Y}} \hat{T} d\chi = \int_{\tilde{Y}} \tilde{T} d\chi.$$

One has

$$\int_{\tilde{Y}} \tilde{T} d\chi = \prod_{\Xi} \left(\sum_{k=0}^{\infty} \chi(S^k \Xi) T_{\Xi}^k \right).$$

Using the well-known equation

$$\sum_{k=0}^{\infty} \chi(S^k Z) t^k = (1-t)^{-\chi(Z)},$$

one gets

$$\int_{\hat{Y}} \tilde{T} d\chi = \prod_{\Xi} (1 - T_{\Xi})^{-\chi(\Xi)}.$$

Let us fix an arbitrary $\underline{V} \in \mathbb{Z}_{\geq 0}^r$ and let us prove Equation (3) up to elements in the ring $K_0((G, r) - \text{sets})$ represented by triples $(X^A, \underline{w}^A, \alpha^A)$ with $\underline{w}^A(x) \leq \underline{V}$ for all $x \in X^A$. This will imply Equation (3) itself. For that we can suppose that the modification $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$ is such that, for any $f \in \mathcal{O}_{\mathbb{C}^2, 0}$ with $\underline{v}(f) \leq \underline{V}$, the strict transform of the curve $\{f = 0\}$ intersects the exceptional divisor \mathcal{D} only at smooth points. This can be achieved by an additional G -invariant series of blowing-ups of intersection points of components of the exceptional divisor. Such additional blowing-ups add, to the stratification $\{\Xi\}$ of $\dot{\mathcal{D}}$, strata with zero Euler characteristics and therefore do not change the right hand side of Equation (3). Let $\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}$ be the set of $f \in \mathcal{O}_{\mathbb{C}^2, 0}$ with $\underline{v}(a^* f) \leq \underline{V}$ for all $a \in G$.

Let $I : \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}} \rightarrow Y$ be the map which sends the class of a function $f \in \mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}$ to the intersection of the strict transform of the zero-level curve $\{f = 0\}$ with the exceptional divisor \mathcal{D} (i.e. to the collection of the intersection points counted with the corresponding multiplicities). One has the commutative diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}} & \xrightarrow{I} & Y \\ p \downarrow & & \downarrow p \\ \mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}/G & \xrightarrow{\hat{I}} & \hat{Y} \end{array}$$

In general, $T \neq \hat{T} \circ \hat{I}$ because the isotropy subgroup of a point in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}$ can be a proper subgroup of the isotropy subgroup of its image in Y and therefore the G -orbits of a point in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}$ and of its image in Y can be different (as G -sets). (If these isotropy subgroups coincide, one has the equality.)

To compute the integral of the function T over $\mathbb{P}\mathcal{O}_{\mathbb{C}^2, 0}^{\underline{V}}/G$, we shall, for each point $\hat{y} \in \hat{Y}$, construct a point $f_{\hat{y}}$ in $\hat{I}^{-1}(\hat{y})$ (i.e. the orbit of a function) so that $T_{f_{\hat{y}}} = \hat{T}(\hat{y})$ and the complement $\hat{I}^{-1}(\hat{y}) \setminus \{f_{\hat{y}}\}$ is fibred into \mathbb{C}^* -families with the function T constant along fibres. This implies that

$$\int_{\hat{I}^{-1}(\hat{y})} T d\chi = \hat{T}(\hat{y})$$

and the Fubini formula applied to the map \widehat{I} gives (up to terms under consideration)

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}^V/G} Td\chi = \int_{\widehat{Y}} \widehat{T}d\chi = \prod_{\Xi} (1 - T_{\Xi})^{-\chi(\Xi)}.$$

Let $\widehat{y} \in \widehat{Y}$ be the orbit of $y = \sum_{i=1}^m x_i \in Y$. Let $f_y := \prod_{i=1}^m h_{x_i}$. The isotropy subgroup of f_y in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ coincides with the isotropy subgroup of y and therefore $T_{f_{\widehat{y}}} = \widehat{T}(\widehat{y})$ (here $f_{\widehat{y}} = p(f_y)$ is the image of f_y in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}/G$).

Let $g \in I^{-1}(y)$. The strict transforms of the curves $\{g = 0\}$ and $\{f_y = 0\}$ intersect the exceptional divisor \mathcal{D} at the same points with the same multiplicities. Therefore the ration $\psi = \frac{g \circ \pi}{f_y \circ \pi}$ of the liftings $g \circ \pi$ and $f_y \circ \pi$ of the functions g and f_y to the space \mathcal{X} of the modification has neither zeros nor poles on the exceptional divisor \mathcal{D} and thus is constant on it. Therefore (multiplying g by a constant) one can assume that the ratio ψ is equal to 1 on \mathcal{D} .

Let $g_{\lambda} := f_y + \lambda(g - f_y)$, $\lambda \in \mathbb{C}^*$. One has $\frac{g_{\lambda} \circ \pi}{f_y \circ \pi} = 1$ on the exceptional divisor \mathcal{D} . Therefore $I(g_{\lambda}) = I(f_y) = y$. Moreover the isotropy subgroup of each g_{λ} in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ coincides with the isotropy subgroup of g and therefore $T_{p(g_{\lambda})}$ is constant. This proves the statement. \square

As above, let \mathbb{C}^2 be the complex plane with a representation of a finite group G . Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be a (generally speaking reducible) plane curve singularity and let $C = \bigcup_{i=1}^r C_i$ be its representation as the union of irreducible components. Let $\varphi_i : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ be a parameterization (uniformization) of the component C_i , i.e., a germ of an analytic map such that $\text{Im } \varphi_i = C_i$ and φ_i is an isomorphism between \mathbb{C} and C_i outside of the origin. For a germ $g \in \mathcal{O}_{\mathbb{C}^2,0}$, let $v_i(g)$ be the degree of the leading term in the power series decomposition of the function $g \circ \varphi_i : (\mathbb{C}, 0) \rightarrow \mathbb{C}$ at the origin:

$$g \circ \varphi_i(\tau) = a \cdot \tau^{v_i(f)} + \text{terms of higher degree}, \quad a \neq 0.$$

(If $g \circ \varphi_i \equiv 0$, one assumes $v_i(g) = +\infty$.) The function $v_i : \mathcal{O}_{\mathbb{C}^2,0} \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is a valuation. The collection v_1, \dots, v_r of the valuations corresponding to the components of the curve C defines a multi-index filtration $\{J(\underline{v})\}$ by (1).

Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$ be a G -invariant embedded resolution of the curve C . This means that \mathcal{X} is a smooth complex surface with an action of the group G commuting with π , π is a proper complex analytic map which is an isomorphism outside of the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$, the total transform

$\pi^{-1}(C)$ of the curve C is a normal crossing divisor on \mathcal{X} (and therefore \mathcal{D} is a normal crossing divisor as well), and moreover, at each intersection point x of two components of the total transform $\pi^{-1}(C)$ of the curve C , each of these components is invariant with respect to the isotropy subgroup $G_x = \{a \in G : ax = x\}$ of the point x . This resolution can be obtained from $(\mathbb{C}^2, 0)$ by a (G -invariant) sequence of blowing-ups at points in the preimage of the origin.

Let $\overset{\circ}{\mathcal{D}}$ be the "smooth" G -invariant part of the exceptional divisor \mathcal{D} in the total transform $\pi^{-1}(C)$ of the curve C , i.e. \mathcal{D} itself minus all intersection points of all the components of $\pi^{-1}(C)$ and of its images under the G -action. Let $\{\Xi\}$ be a stratification of $\overset{\circ}{\mathcal{D}}/G$ defined in the same way as above (for divisorial valuations). (In fact one can simply take the intersections of the strata of the stratification for the divisorial case with $\overset{\circ}{\mathcal{D}}/G$.) Let $T_{\Xi} \in K_0((G, r) - \text{sets})$ be defined as above. The same arguments give the following statement.

Theorem 2

$$P_{\{v_i\}}^G = \prod_{\Xi} (1 - T_{\Xi})^{-\chi(\Xi)} .$$

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